



Barber, B. (2014). Maximum Hitting for n Sufficiently Large. *Graphs and Combinatorics*, 30(2), 267-274. <https://doi.org/10.1007/s00373-012-1281-9>

Peer reviewed version

Link to published version (if available):
[10.1007/s00373-012-1281-9](https://doi.org/10.1007/s00373-012-1281-9)

[Link to publication record in Explore Bristol Research](#)
PDF-document

This is the accepted author manuscript (AAM). The final published version (version of record) is available online via Springer Verlag at <http://dx.doi.org/10.1007/s00373-012-1281-9>. Please refer to any applicable terms of use of the publisher.

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available:
<http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/>

Maximum hitting for n sufficiently large

Ben Barber*

November 30, 2012

Abstract

For a left-compressed intersecting family $\mathcal{A} \subseteq [n]^{(r)}$ and a set $X \subseteq [n]$, let $\mathcal{A}(X) = \{A \in \mathcal{A} : A \cap X \neq \emptyset\}$. Borg asked: for which X is $|\mathcal{A}(X)|$ maximised by taking \mathcal{A} to be all r -sets containing the element 1? We determine exactly which X have this property, for n sufficiently large depending on r .

1 Introduction

Write $[n] = \{1, 2, \dots, n\}$ and $[m, n] = \{m, m+1, \dots, n\}$. Denote the set of r -sets from a set S by $S^{(r)}$. A *family* of sets is a subset of $[n]^{(r)}$ for some n and r . We think of a set A as an increasing sequence of elements $a_1 a_2 \dots a_r$. The *compression order* on $[n]^{(r)}$ has $A \leq B$ if and only if $a_i \leq b_i$ for $1 \leq i \leq r$. A family \mathcal{A} is *left-compressed* if $A \in \mathcal{A}$ whenever $A \leq B$ for some $B \in \mathcal{A}$. The corresponding notion of left-compression is described in Section 2.

We call a family *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. (If $n < 2r$ then every family is intersecting.) The most basic result about intersecting families is the Erdős-Ko-Rado Theorem. For any n and r , write $\mathcal{S} = \{A \in [n]^{(r)} : 1 \in A\}$ for the *star* at 1.

Theorem 1 (Erdős-Ko-Rado [3]). *If $n \geq 2r$ and $\mathcal{A} \subseteq [n]^{(r)}$ is intersecting, then $|\mathcal{A}| \leq |\mathcal{S}|$.*

Borg considered a variant problem where we only count members that meet some fixed set X . For a family \mathcal{A} and a non-empty set X , write

$$\mathcal{A}(X) = \{A \in \mathcal{A} : A \cap X \neq \emptyset\}.$$

Theorem 1 tells us that we can maximise $|\mathcal{A}(X)|$ by taking \mathcal{A} to consist of all r -sets containing some fixed element of X . To avoid this trivial case we insist that \mathcal{A} be left-compressed, which rules out stars centred anywhere but 1. The star at 1 remains the optimal family if $1 \in X$, so we assume further that $X \subseteq [2, n]$.

Question 2. *For which X do we have $|\mathcal{A}(X)| \leq |\mathcal{S}(X)|$ for all left-compressed intersecting families \mathcal{A} ?*

Borg asked this question in [2], giving a complete answer for the case $|X| \geq r$ and a partial answer for the case $|X| < r$. Call X *good* (for n and r) if for every left-compressed intersecting family $\mathcal{A} \subseteq [n]^{(r)}$ we have $|\mathcal{A}(X)| \leq |\mathcal{S}(X)|$.

*Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK. b.a.barber@dpmmms.cam.ac.uk

Theorem 3 (Borg [2]). *Let $r \geq 2$, $n \geq 2r$ and $X \subseteq [2, n]$.*

- (a) *If $|X| > r$, then X is good.*
- (b) *If X is good and $X \leq X'$, then X' is good.*
- (c) *For any $k \leq r$, $\{2k, 2k+2, \dots, 2r\}$ is good.*
- (d) *If $n = 2r$ and $|X| = r$, then X is good if and only if $\{2, 4, \dots, 2r\} \leq X$.*
- (e) *If $n > 2r$, $|X| = r$ and either*
 - (i) *$r \geq 4$ and $X \neq [2, r+1]$,*
 - (ii) *$r = 3$ and $\{2, 3\} \not\subseteq X$, or*
 - (iii) *$r = 2$ and $\{2, 3\} \neq X$,**then X is good. Otherwise, X is not good.*

It is not true that all X are good. For example, consider the *Hilton-Milner* family $\mathcal{T} = \mathcal{S}([2, r+1]) \cup \{[2, r+1]\}$. The family \mathcal{T} is left-compressed and for any $X \subseteq [2, r+1]$, $|\mathcal{T}(X)| = |\mathcal{S}(X)| + 1$, so X is not good.

Our main result is that, surprisingly, for large n and $|X| \geq 4$ this turns out to be the only obstruction.

Theorem 4. *Let $r \geq 3$, $n \geq 2r$ and $X \subseteq [2, n]$ with $|X| \leq r$. If $X \not\subseteq [2, r+1]$ and either*

- (i) $|X| \geq 4$,
- (ii) $|X| = 3$ and $\{2, 3\} \not\subseteq X$,
- (iii) $|X| = 2$ and $2, 3 \notin X$, or
- (iv) $|X| = 1$,

then, for n sufficiently large, X is good. Otherwise, X is not good.

For $r = 2$, condition (iii) needs to be replaced by $X \neq \{2, 3\}$. The result can then be checked easily by hand or read out of Theorem 3 in conjunction with the Hilton-Milner example, so we assume $r \geq 3$ for simplicity.

Our proof uses Ahlswede and Khachatrian's notion of generating sets to express the sizes of maximal left-compressed intersecting families, and their restrictions under X , as polynomials in n . It turns out to be sufficient to consider only leading terms, reducing a question about intersecting families of r -sets to a question about intersecting families of 2-sets, which have a very simple structure.

Section 2 sets out the basic properties of compressions and generating sets that we shall use. Section 3 describes a way of thinking about maximal left-compressed intersecting families and proves the lemma that allows us to compare coefficients of polynomials instead of set sizes. Section 4 completes the proof of Theorem 4. Section 5 discusses possible improvements and generalisations.

2 Compressions and generating sets

In this section we describe the notion of left-compression corresponding to \leq on $[n]^{(r)}$ and the use of generating sets.

2.1 Compressions

For a set A , and $i < j$, the ij -compression of A is

$$C_{ij}(A) = \begin{cases} A - j + i & \text{if } j \in A, i \notin A, \\ A & \text{otherwise;} \end{cases}$$

that is, replace j by i if possible. Observe that $A \leq B$ if and only if A can be obtained from B by a sequence of ij -compressions.

For a set family \mathcal{A} , define

$$C_{ij}(\mathcal{A}) = \{C_{ij}(A) : A \in \mathcal{A} \text{ and } C_{ij}(A) \notin \mathcal{A}\} \cup \{A : A \in \mathcal{A} \text{ and } C_{ij}(A) \in \mathcal{A}\};$$

that is, compress A if possible. Observe that \mathcal{A} is left-compressed if and only if $C_{ij}(\mathcal{A}) = \mathcal{A}$ for all $i < j$. We will use the following basic result.

Lemma 5. *If \mathcal{A} is intersecting then $C_{ij}(\mathcal{A})$ is intersecting.*

Proof. The proof is an easy case check. Details, and a further introduction to compressions, can be found in Frankl's survey article [4]. \square

Lemma 5 means that we can always compress an intersecting family to a left-compressed intersecting family of the same size by repeatedly applying ij -compressions. We eventually reach a left-compressed family as $\sum_{A \in \mathcal{A}} \sum_{i=1}^r a_i$ is positive and strictly decreases with each successful compression.

2.2 Generating sets

For any r and n , and a collection \mathcal{G} of sets, the family *generated* by \mathcal{G} is

$$\mathcal{F}(r, n, \mathcal{G}) = \{A \in [n]^{(r)} : A \supseteq G \text{ for some } G \in \mathcal{G}\}.$$

Generating sets were introduced by Ahlswede and Khachatrian [1], and are useful for the study of intersecting families because they give a restricted number of sets on which all the intersecting actually happens.

Lemma 6 ([1]). *For $n \geq 2r$, $\mathcal{F}(r, n, \mathcal{G})$ is intersecting if and only if \mathcal{G} is.*

Proof. If \mathcal{G} is intersecting then certainly $\mathcal{F}(r, n, \mathcal{G})$ is. Conversely, if \mathcal{G} contains two disjoint sets then (since $n \geq 2r$) they can be completed to disjoint r -sets in $\mathcal{F}(r, n, \mathcal{G})$. \square

If \mathcal{G} generates a left-compressed intersecting family then

$$\mathcal{G}' = \{G' : G' \leq G \text{ for some } G \in \mathcal{G}\}$$

generates the same family, so we may assume that \mathcal{G} is 'left-compressed' (overlooking non-uniformity) and can therefore be described by listing its maximal elements. It is convenient to take

$$\mathcal{F}(r, n, \mathcal{G}) = \{A \in [n]^{(r)} : A \prec G \text{ for some } G \in \mathcal{G}\},$$

where $A \prec G$ (' A is generated by G ') if and only if $|G| \leq |A|$ and $a_i \leq g_i$ for $1 \leq i \leq |G|$. We can think of \prec as an extension of \leq to the non-uniform case, where 'missing' elements are assumed to take the value ∞ . Thus

$$\begin{aligned} 123 &\prec 12 \quad (= 12\infty); \\ (12\infty =) 12 &\not\prec 123. \end{aligned}$$

The following weaker form of Lemma 6 is better suited to our new definition and is sufficient for our purposes.

Corollary 7. *Let $n \geq 2r$ and \mathcal{G} be a collection of subsets of $[2s]$ of size at most s . If $\mathcal{F}(s, 2s, \mathcal{G})$ is intersecting, then so is $\mathcal{F}(r, n, \mathcal{G})$.* \square

3 Maximal left-compressed intersecting families

We say an intersecting family $\mathcal{A} \subseteq [n]^{(r)}$ is *maximal* if no other set can be added to \mathcal{A} while preserving the intersecting property. The maximal objects in the set of left-compressed intersecting families are maximal intersecting families (otherwise an extension could be compressed to a left-compressed extension), so the ordering of 'maximal' and 'left-compressed' is unimportant.

The maximal left-compressed intersecting subfamilies of $[n]^{(2)}$ are $\{12, 13, \dots, 1n\}$ and $\{12, 13, 23\}$, and we can already distinguish between these families when $n = 4$. In fact, the same phenomenon occurs for all r .

Lemma 8. *Let $\mathcal{A} \subseteq [2r]^{(r)}$ be a maximal left-compressed intersecting family and $n \geq 2r$. Then \mathcal{A} extends uniquely to a maximal left-compressed intersecting subfamily of $[n]^{(r)}$. Moreover, every maximal left-compressed intersecting subfamily of $[n]^{(r)}$ arises in this way.*

Proof. Since \mathcal{A} is left-compressed, it can be completely described by listing its \leq -maximal elements A_1, \dots, A_k . Some of these sets might contain final segments of $[2r]$. The idea is that the elements of these final segments would take larger values if they were allowed to, so we obtain a generating set by 'replacing them by ∞ '.

For $A = A_i$, take s greatest with $a_s < r + s$ (s exists since $[r + 1, 2r]$ is not a member of any left-compressed intersecting family), and let $A' = a_1 \dots a_s$. Then $\mathcal{G} = \{A'_1, \dots, A'_k\}$ generates \mathcal{A} , as the sets generated by A'_i are precisely those lying below A_i . Since \mathcal{G} is a collection of subsets of $[2r]$ of size at most r and $\mathcal{A} = \mathcal{F}(r, 2r, \mathcal{G})$ is intersecting, Corollary 7 tells us that $\mathcal{F}(r, n, \mathcal{G})$ is a left-compressed intersecting family for every n .

Now let \mathcal{B} be any extension of \mathcal{A} to a left-compressed intersecting subfamily of $[n]^{(r)}$. We will show that $\mathcal{B} \subseteq \mathcal{F}(r, n, \mathcal{G})$. Indeed, if $\mathcal{B} \not\subseteq \mathcal{F}(r, n, \mathcal{G})$ then there is a $B \in \mathcal{B} \setminus \mathcal{F}(r, n, \mathcal{G})$. We claim that there is a $B' \in [2r]^{(r)}$ with $B' \leq B$ and $B' \notin \mathcal{F}(r, 2r, \mathcal{G})$, contradicting the maximality of \mathcal{A} .

We obtain B' from B by compressing as little as possible to get $B' \subseteq [2r]$; that is, we take $B' = (B \cap [2r]) \cup [q, 2r]$ with q chosen such that $|B'| = r$. Explicitly, $b'_i = \min(b_i, r + i)$. Now take $G \in \mathcal{G}$. Since $B \notin \mathcal{F}(r, n, \mathcal{G})$, there is an i with $b_i > g_i$. By construction, $r + i > g_i$. So $b'_i = \min(b_i, r + i) > g_i$, and G does not generate B' . Hence \mathcal{A} extends uniquely to a maximal left-compressed intersecting subfamily of $[n]^{(r)}$.

It remains to show that every maximal left-compressed intersecting subfamily of $[n]^{(r)}$ arises in this way. So suppose $\mathcal{C} \subseteq [n]^{(r)}$ is a maximal left-compressed intersecting family with $\mathcal{C} \cap [2r]^{(r)}$ not maximal. Let \mathcal{D}_0 be an extension of $\mathcal{C} \cap [2r]^{(r)}$ to a maximal left-compressed intersecting subfamily of $[2r]^{(r)}$, and let \mathcal{D} be the unique maximal extension of \mathcal{D}_0 to $[n]^{(r)}$. Since \mathcal{C} is maximal and $\mathcal{D} \setminus \mathcal{C} \neq \emptyset$, there is a $C \in \mathcal{C} \setminus \mathcal{D}$. As above, we obtain $C' \in [2r]^{(r)}$ with $C' \leq C$ and $C' \notin \mathcal{D}_0$. But then $C' \notin \mathcal{C}$, contradicting the assumption that \mathcal{C} is left-compressed. \square

Lemma 8 allows a compact description of maximal left-compressed intersecting families. For example, $\{1\}$ generates the star and $\{1(r+1), [2, r+1]\}$ generates the Hilton-Milner family. Enumerating the generating sets using a computer is feasible for small r ; for $r = 3$ they are $\{1\}, \{23\}, \{345\}, \{14, 234\}, \{13, 235, 145\}$ and $\{12, 245\}$.

In view of Lemma 8, our key tool is the following.

Lemma 9. *Let $n \geq 2$, $X \subseteq [2, 2r]$. Then*

$$|\mathcal{F}(r, n, \mathcal{G})(X)| = \sum_{i=1}^r |\mathcal{F}(i, 2r, \mathcal{G})(X)| \binom{n-2r}{r-i}.$$

Proof. How do we construct a member of $\mathcal{F}(r, n, \mathcal{G})(X)$? We first choose an initial segment for our set that is contained in $[2r]$ and witnesses the membership of $\mathcal{F}(r, n, \mathcal{G})(X)$ (i.e. meets X and is \prec some $G \in \mathcal{G}$). We then complete our set by taking as many elements as we need from outside $[2r]$. This gives rise to the size claimed. \square

4 Proof of Theorem 4

We first show that X is not good if the given conditions do not hold. We have already seen that for $X \subseteq [2, r+1]$ the Hilton-Milner family shows that X is not good for any n . In each of the remaining cases we claim that the family generated by $\{23\}$ shows that X is not good for any n .

So take $X = 23k$ with $k \geq r+2$. We have

$$|\mathcal{F}(r, n, \{1\})(23k)| = \binom{n-2}{r-2} + \binom{n-3}{r-2} + \binom{n-4}{r-2},$$

where the first term counts the sets containing 1 and 2, the second term the sets containing 1 and 3 but not 2, and the third term the sets containing 1 and k but neither 2 nor 3. Similarly,

$$|\mathcal{F}(r, n, \{23\})(23k)| = \binom{n-2}{r-2} + \binom{n-3}{r-2} + \binom{n-3}{r-2},$$

where the terms count the sets containing 1 and 2, the sets containing 1 and 3 but not 2, and the sets containing 2 and 3 but not 1 respectively. Since $r \geq 3$, $|\mathcal{F}(r, n, \{23\})(23k)| > |\mathcal{F}(r, n, \{1\})(23k)|$ and $23k$ is not good.

Next take $X = 3j$ with $j \geq r+2$. We have

$$|\mathcal{F}(r, n, \{1\})(3j)| = \binom{n-2}{r-2} + \binom{n-3}{r-2},$$

where the terms count the sets containing 1 and 3, and the sets containing 1 and j but not 3 respectively. Similarly,

$$|\mathcal{F}(r, n, \{23\})(3j)| = \binom{n-2}{r-2} + \binom{n-3}{r-2} + \binom{n-4}{r-3},$$

where the terms count the sets containing 1 and 3, the sets containing 2 and 3 but not 1, and the sets containing 1, 2 and j but not 3 respectively. Again, since $r \geq 3$, $|\mathcal{F}(r, n, \{23\})(3j)| > |\mathcal{F}(r, n, \{1\})(3j)|$ and $3j$ is not good. It follows from Theorem 3(b) that $2j$ is not good either.

Now we take X satisfying the conditions of the theorem and show that X is good for n sufficiently large. We will show that, for any $\mathcal{G} \neq \{1\}$, $|\mathcal{F}(2, 2r, \mathcal{G})(X)| < |\mathcal{F}(2, 2r, \{1\})(X)| = |X|$. Note that, for any \mathcal{G} , $|\mathcal{F}(1, 2r, \mathcal{G})(X)| = 0$ as the only possible singleton generator is 1, which does not meet X . So by Lemma 9, $\mathcal{F}(2, n, \mathcal{G})(X)$ has size polynomial in n with leading coefficient $|\mathcal{F}(2, 2r, \mathcal{G})(X)|$, from which the result will follow.

There are two maximal left-compressed intersecting families of 2-sets, and $\mathcal{F}(2, 2r, \mathcal{G})(X)$ must be contained in one of them. We handle each case separately.

Suppose first that $\mathcal{F}(2, 2r, \mathcal{G})(X) \subseteq \{12, 13, 23\}$. Then it is enough to show that

$$|\{12, 13, 23\}(X)| < |X|.$$

This is clearly true for $|X| \geq 4$. If $|X| = 3$, then it is true because one of 2 or 3 is missing from X so that $|\{12, 13, 23\}(X)| \leq 2$. If $|X| = 2$, then it is true because both 2 and 3 are missing from X , so that $|\{12, 13, 23\}(X)| = 0$. Finally, if $|X| = 1$, then it is true because $X = \{i\}$ with $i \geq r+2 \geq 4$.

Next suppose that $\mathcal{F}(2, 2r, \mathcal{G})(X) \subseteq \{12, 13, \dots, 1(2r)\}$. Since $\mathcal{F}(r, 2r, \mathcal{G})$ is left-compressed and has a member not containing the element 1, it has $[2, r+1]$ as a member. Hence by the intersecting property of the generators, $\mathcal{F}(2, 2r, \mathcal{G})(X)$ cannot contain $1j$ for any $j \geq r+2$. But $X \not\subseteq [2, r+1]$, so there is such a $j \in X \setminus [2, r+1]$ and $|\mathcal{F}(2, 2r, \mathcal{G})(X)| < |X|$. \square

5 Improvements and generalisations

What happens for small n ? Theorem 3(c) tells us that our characterisation cannot be correct for all $n \geq 2r$.

Question 10. *How large is ‘sufficiently large’ for n in Theorem 4?*

For $2 \leq r \leq 5$, computational results suggest that $n \geq 2r+2$ is large enough for our characterisation to be correct. It would be particularly nice to show that $n \geq 2r+c$ is sufficient for some constant c independent of r .

A natural conjecture is that for $n = 2r$, $[2k, 2k+2, \dots, 2r]$ is the unique minimal good set of its size. However, this is false; computational results give that $\{7, 10\}$ and $\{5, 8, 10\}$ are unique minimal good sets of their size when $r = 5$.

Question 11. *Is there a ‘nice’ characterisation of the good sets for $n = 2r$ when r is sufficiently large?*

It seems unlikely that a good explicit description exists for intermediate values of r and n . The following may be easier.

Question 12. *Is there a short list of families, one of which maximises $|\mathcal{A}(X)|$ for any X ?*

Versions of Lemma 8 hold for any property that is preserved under left-compression and can be detected on generating sets. The most obvious candidate is that of being t -intersecting (a family \mathcal{A} is t -intersecting if $|A \cap B| \geq t$ for all $A, B \in \mathcal{A}$). Indeed, an identical argument gives the corresponding result that, for large n , a set $X \subseteq [t+1, n]$ with $|X| \geq t+3$ is good if and only if $X \not\subseteq [t+1, r+1]$. (For smaller X the form of good X is again decided by the need to prevent problems caused when $\mathcal{F}(t+1, 2r-t+1, \mathcal{G})(X) \subseteq [t+2]^{(t+1)}$.)

In the context of t -intersecting families it may be more natural to consider

$$\mathcal{A}(s, X) = \{A \in \mathcal{A} : |A \cap X| \geq s\}.$$

For $s = 1$ the argument relies on the fact that maximal left-compressed t -intersecting families of $(t+1)$ -sets have one of two very simple forms. For $s = 2$, even the $t = 1$ case is complicated by the larger number of structures of intersecting families of 3-sets (more generally, $(t+s)$ -sets); this problem seems likely to get worse for larger s and t .

Acknowledgements. I would like to thank the anonymous referees for carefully reading an earlier draft of this paper and making a number of helpful comments.

References

- [1] Rudolf Ahlswede and Levon H. Khachatrian. The complete intersection theorem for systems of finite sets. *European J. Combin.*, 18(2):125–136, 1997.
- [2] Peter Borg. Maximum hitting of a set by compressed intersecting families. *Graphs and Combinatorics*, 27(6):785–797, 2011.
- [3] P. Erdős, Chao Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.
- [4] P. Frankl. The shifting technique in extremal set theory. In C. Whitehead, editor, *Surveys in Combinatorics*, volume 123 of *London Math. Soc. Lecture Notes Series*, pages 81–110. Cambridge University Press, Cambridge, 1987.